Then Date: ________

$U(f,P)-L(f,P)$

$$= \sum_{i=1}^{n} \left(\sup_{[x_{i-1},x_i]} f(x) - \inf_{[x_{i-1},x_i]} f(x) \right)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \left(f(x_{i-1}) - f(x_i) \right) \frac{b-a}{n}$$

$$= \frac{b-a}{n} \Big(f(x_0) - \cancel{f(x_1)} + \cancel{f(x_1)} - \cancel{f(x_2)} + \cancel{f(x_2)} - \cancel{f(x_3)} + \cancel{f(x_3)} - f(x_4) + \cdots - f(x_n) \Big)$$

$$= \frac{b-a}{n} \left(f(x_0) - f(x_n) \right) = \frac{(b-a)(f(a)-f(b))}{n}.$$

Choose n to be greater than

$$\frac{(b-a)(f(a)-f(b))}{\varepsilon}$$

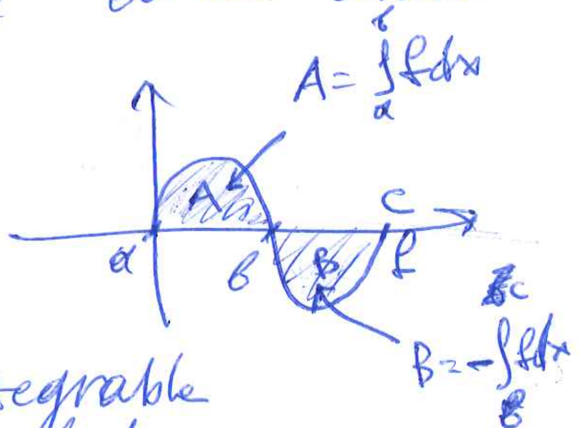
Then

$$U(f,P) - L(f,P) < \frac{\cancel{(b-a)(f(a)-f(b))}\,\varepsilon}{\cancel{(b-a)(f(a)-f(b))}} = \varepsilon.$$

This means f is integrable. $\square$

<u>Note.</u> If f is in(de)creasing, then f is bounded on $[a,b]$.

Remark. One interpretes the Riemann integral as the signed area under the graph.



Theorem. 1) If f is ~~bounded~~ integrable on $[a,b]$ and $k \in \mathbb{R}$, then kf is integrable and

$$\int_a^b kf\,dx = k\int_a^b f\,dx.$$

2) If f, g are integrable on $[a,b]$, then so is $f+g$, and

$$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx.$$

3) If f, g integrable and $f \leq g$ on $[a,b]$, then $\int_a^b f\,dx \leq \int_a^b g\,dx$.

4) If f is integrable on $[a,b]$ and $[b,c]$, then it is integr. on $[a,c]$ and

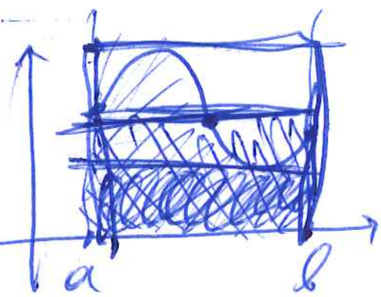
$$\int_a^c f\,dx = \int_a^b f\,dx + \int_b^c f\,dx.$$

5) If f is integrable on $[a,b]$, then so is $|f|$ and
$$\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx.$$
$\square$

<u>Proof.</u> Straightforward.

<u>Theorem</u> (Mean value theorem (MVT)).
If f is continuous on $[a,b]$, then there exists $c \in [a,b]$ s.t.
$$\int_a^b f(x)\,dx = f(c)(b-a).$$

<u>Proof.</u>



If f is constant on $[a,b]$, then the statement is obvious.
Assume it's not constant. Denote $m = \inf_{[a,b]} f(x)$, $M = \sup_{[a,b]} f(x)$. Because f is contin., there are $x_m, x_M \in [a,b]$ s.t. $f(x_m) = m$ and $f(x_M) = M$. We may assume $x_m < x_M$. Then apply IVT

on $[x_m, x_M]$.

Observe than

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Then

$$\frac{\int_a^b f\,dx}{b-a} \in [m, M].$$

Now, $f(x_m) = m$ and $f(x_M) = M$, so IVT implies the existence of $c \in [x_m, x_M]$ s.t.

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a}, \quad \text{or}$$

$$\int_a^b f\,dx = f(c)(b-a). \qquad \square$$

<u>Def.</u> F is ~~the~~ an antiderivative (a primitive) of f if $F' = f$.

<u>Theorem</u> (Fundamental theorem of calculus). Assume $f: [a,b] \to \mathbb{R}$ is <u>continuous</u>. Define $F(x) = \int_a^x f(t)\,dt$. Then F is an antiderivative of f and

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

<u>Proof.</u>

<u>Note.</u> F is well-defined because f is continuous (and thus integrable) on $[a, x]$ for each $x \in [a,b]$.

<u>Proof.</u> We claim $F' = f$ on (a,b).

Take $x \in (a,b)$ and $h \in (0, b-x)$.

Motivation: ~~Take~~ $F'(x) = \lim\limits_{h \to 0} \dfrac{F(x+h) - F(x)}{h}$.

Consider

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt = f(c)\big((\cancel{x}+h) - \cancel{x}\big) = f(c)\,h$$

for some $c \in [x, x+h]$ by the MVT.

Now, f is unif. continuous on $[a,b]$. For each $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$ (here, $x, y \in (a,b)$).

If $|h| < \delta$, then $|c - x| < \delta$ and

$|f(c) - f(x)| < \varepsilon$. Therefore,

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right|$$

$$= \left| \frac{f(c)h}{h} - f(x) \right| = |f(c) - f(x)| < \varepsilon.$$

This means

$$\lim_{h \to 0+} \frac{F(x+h) - F(x)}{h} = f(x)$$

Similarly, $\displaystyle\lim_{h \to 0-} \frac{F(x+h) - F(x)}{h} = f(x)$.

Thus, $F'(x) = f(x)$. $\square$

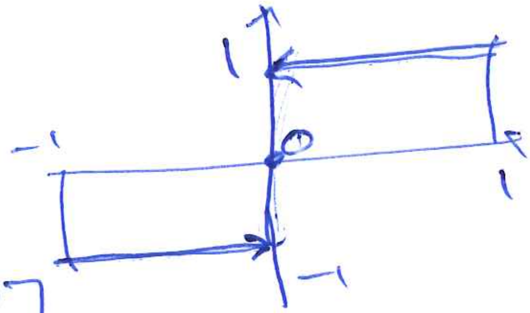
<u>Note.</u> If $G'(x) = f(x)$ on (a, b), then $(G - F)' = f - f = 0$, so $G - F = c$, so $G = F + c$, where $c \in \mathbb{R}$.

Also, $\displaystyle\int_a^b f(x)\,dx = \overset{F(b)+c - F(a)+c}{G(b) - G(a)}$, provided G is continuous on $[a, b]$.

Remark. If f is discontinuous, then $F(x)=\int_a^x f(t)\,dt$ might not be differentiable.

Example. Take $f=\operatorname{sign} x$ on $[-1,1]$.

$$f(x)=\begin{cases}-1, & x<0,\\ 0, & x=0,\\ 1, & x>0,\end{cases} \quad \text{on } [-1,1].$$



Then

$$\int_{-1}^{x} f(t)\,dt = |x| \quad \text{on } [-1,1].$$

Theorem (integration by parts).
If $u, v : [a,b]\to\mathbb{R}$ are differentiable on (a,b) and u', v' are continuous on (a,b), then

$$\int_a^b u v'\,dx = uv\Big|_a^b - \int u' v\,dx.$$

Proof. The product rule yields

$$(uv)' = u'v + uv'.$$

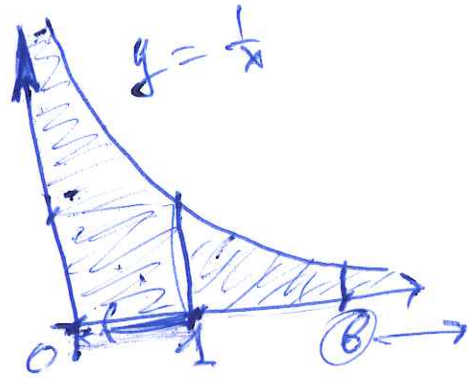
Integrate this from a to b:

$$\int_a^b (uv)'\,dx = \int_a^b uv'\,dx + \int_a^b u'v\,dx,$$

$$uv\Big|_a^b = \int_a^b uv'\,dx + \int_a^b u'v\,dx,$$

$$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx. \quad \square$$

Improper integrals.



Consider a function $f: (a,b] \to \mathbb{R}$, not necessarily bounded. Assume f is integrable on $[a+\varepsilon, b]$ for all $\varepsilon \in (0, b-a)$.

Then

$$\lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b f(x)\,dx$$

is called the improper integral of f on $[a,b]$ (of 1st kind).

This is usually denoted $\int_a^b f(x)\,dx$.

Assume $f : [a,\infty) \to \mathbb{R}$ is integrable on $[a,b]$ for all $b > a$.

Then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

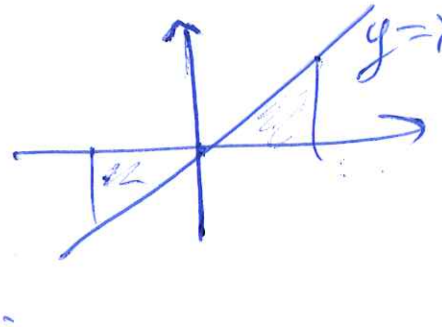
is called the <u>improper integral</u> (of the 2^{nd} kind) of f on $[a,\infty)$.

We may also define

$$\int_{-\infty}^{\infty} f(x)\,dx = \cancel{\lim} \int_{-\infty}^{0} f(x)\,dx + \int_0^\infty f(x)\,dx,$$

provided both integrals on the right exist.

$$\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{a\to\infty} \int_{-a}^{a} f(x)\,dx.$$



For example,

$\int_{-\infty}^{\infty} x\,dx$ does not <u>exist</u>.

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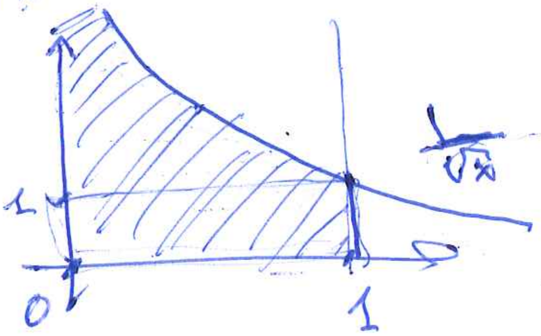
Example. ① Compute $\int_0^1 \frac{1}{\sqrt{x}}\,dx$.



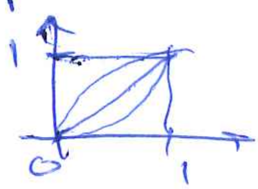
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{\varepsilon\to 0+} \int_\varepsilon^1 \frac{dx}{\sqrt{x}}$$

$$= \lim_{\varepsilon\to 0+} 2\sqrt{x}\,\Big|_\varepsilon^1 = \lim_{\varepsilon\to 0+} 2(\sqrt{1}-\sqrt{\varepsilon}) = 2.$$

② More generally: $\int_0^1 x^p\,dx$, $p \in \mathbb{R}$.

a) $p \geq 0$. Then x^p is contin. on $[0,1]$, so it is integrable. This case is easy.



b) $p \in (-1, 0)$.

$$\int_0^1 x^p\,dx = \lim_{\varepsilon\to 0+} \int_\varepsilon^1 x^p\,dx = \frac{1}{p+1} \lim_{\varepsilon\to 0+} x^{p+1}\Big|_\varepsilon^1$$

$$= \frac{1}{p+1} \lim_{\varepsilon\to 0+} (1 - \varepsilon^{p+1}) = \frac{1}{p+1}.$$

c) $p < -1$. Same argument implies the integral diverges.